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# HERMITE POLYNOMIALS AND A DUALITY RELATION FOR MATCHINGS POLYNOMIALS

by

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Let $G$ be a graph on $n$ vertices. A $k$-matching in $G$ is a set of $k$ independent edges. If $2k=n$ then a $k$-matching is called perfect. The number of $k$-matchings in $G$ is $p(G, k)$. (We set $p(G, 0)=1$). The matchings polynomial of $G$ is

$$\alpha(G, x) = \sum_{k=0}^{[n/2]} (-1)^k p(G, k)\, x^{n-2k}.$$

Our main result is that the number of perfect matchings in the complement of $G$ is equal to

$$(1) \qquad (2\pi)^{-1/2} \int_{-\infty}^{\infty} \alpha(G, x) \exp(-x^2/2)\, dx.$$

Let $K_m$ be the complete graph on $m$ vertices. Then $\alpha(K_m, x)$ is the Hermite polynomial $\mathrm{He}_n(x)$ of degree $n$. Using (1) we show, amongst other results, that

$$\alpha(\bar{G}, x) = \sum_{k=0}^{[n/2]} p(G, k)\alpha(K_{n-2k}, x).$$

## 1. The basic result

**1.1.** Let $G$ be a graph with vertex set $\{1, \ldots, n\}$. A *k-matching* in $G$ is a set of $k$ edges of $G$, no two of which have a vertex in common. Clearly $k \leqq n/2$, if $k=n/2$ then a $k$-matching of $G$ is called *perfect*. The number of $k$-matchings in $G$ will be denoted by $p(G, k)$, with the convention that $p(G, 0)=1$.

The matchings polynomial $\alpha(G)=\alpha(G, x)$ is

$$\sum_{k=0}^{[n/2]} (-1)^k p(G, k) x^{n-2k}$$

This polynomial was first studied, under the notation $Q(G, x)$, by Heilmann and Lieb in a paper on statistical physics [4]. It arose there as a form of thermodynamic partition function. Many interesting mathematical results on the matchings polynomial can be found in [4]. A more recent survey is presented in [3].

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The complete graph on $n$ vertices will be denoted by $K_n$. The matchings polynomial $\alpha(K_n)$ is identical with the Hermite polynomial $He_n(x)$ of degree $n$. (This latter polynomial is usually defined by

$$He_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}.$$

For further information on Hermite polynomials we refer the reader to [2]. The fact that $He_n(x)=\alpha(K_n, x)$ is proved in [4].)

We use $\bar{G}$ to represent the complement of the graph $G$. The main result of this paper is the following connection between $\alpha(G, x)$ and $\alpha(\bar{G}, x)$.

**1.2. Theorem.** *Let $G$ be a graph. Then the number of perfect matchings in $\bar{G}$ is equal to*

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} \alpha(G, x) \exp(-x^2/2)\, dx.$$

**Proof.** We represent the integral above by $I(G)$. Let $E_n$ denote the graph with $n$ vertices and no edges. Clearly the complement of $E_n$ is $K_n$, hence the number of perfect matchings in $\bar{E}_n$ is zero if $n$ is odd and equals $(n-1)(n-3)\dots 1$ when $n$ is even. It is now a simple exercise in integration by parts to show that the theorem holds when $G=E_n$.

We now prove the theorem for an arbitrary graph $G$ with at least one edge. Assume inductively that our result holds for each proper subgraph of $G$. Suppose $u=\{1, 2\}$ is an edge in $G$. Let $G_u$ be the graph obtained by deleting the edge $\{1, 2\}$, let $G_{12}$ be obtained by deleting the vertices 1 and 2 from $G$.

Trivially, the $k$-matching in $G$ may be partitioned into those that contain $u$ and those that do not. Hence we have

$$p(G, k) = p(G_u, k)+p(G_{12}, k-1). \tag{1}$$

Using the definition of $\alpha(G)$ it follows that

$$\alpha(G, x) = \alpha(G_u, x)-\alpha(G_{12}, x). \tag{2}$$

From (1) we also have

$$p(\bar{G}_u, k) = p(\bar{G}, k)+p(\bar{G}_{12}, k-1), \tag{3}$$

where it is to be understood that complementation is carried out after deletion.

By (2) and the linearity of integration,

$$I(G) = I(G_u)-I(G_{12}). \tag{4}$$

If $n$ is odd then $G_u$ and $G_{12}$ have no perfect matchings, the right side of (4) is zero, by our induction hypothesis. Hence $I(G)=0$. Since $\bar{G}$ also has no perfect matchings when $n$ is odd, the theorem holds in this case.

Suppose then that $n=2m$. As the theorem holds for $G_u$ and $G_{12}$, (4) yields

$$I(G) = p(\bar{G}_u, m)-p(\bar{G}_{12}, m-1),$$

whence the theorem following using (3). ∎

## 2. Some consequences of Theorem 1.2

**2.1.** In this section we will derive some consequences of Theorem 1.2. If $G$ and $H$ are graphs we use $G \cup H$ to denote their disjoint union. It is easy to show that $\alpha(G \cup H) = \alpha(G)\alpha(H)$.

**2.2. Corollary.** *Let $G$ be a graph with $n$ vertices. Then*

$$(n-2k)!\,p(\bar{G}, k) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \alpha(G, x)\alpha(K_{n-2k}, x) \exp(-x^2/2)\, dx.$$

**Proof.** Let $m = n-2k$. The integral given is just $I(G \cup K_m)$. By 1.2 this equals the number of perfect matchings in the complement of $\overline{G \cup K_m}$. Each $k$-matching in $\bar{G}$ can be extended to give $m!$ perfect matchings in $\overline{G \cup K_m}$. On the other hand each perfect matching of $G \cup K_m$ restricts to a $k$-matching of $\bar{G}$, so our result follows. ∎

Taking $G = K_n$ and $m = n-2k$ in 2.2 yields the identity

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} \alpha(K_m)\alpha(K_n) \exp(-x^2/2)\, dx = \begin{cases} m! & m = n, \\ 0 & m \neq n. \end{cases}$$

This is, of course, just a statement of the orthogonality of the Hermite polynomials on the real line, with respect to the weight function $(2\pi)^{-1/2} \exp(-x^2/2)$. It should be evident that 1.2 and 2.2 enable a combinatorial interpretation of many properties of the Hermite polynomials.

One consequence of the orthogonality of the Hermite polynomials is the following.

**2.3. Corollary.** *Let $G$ be a graph with $n$ vertices. Then*

$$\alpha(\bar{G}, x) = \sum_{k=0}^{[n/2]} p(G, k)\alpha(K_{n-2k}, x).$$

**Proof.** It is easy to see that the Hermite polynomials form a basis for the vector space of all polynomials with real coefficients. Hence $\alpha(\bar{G}, x)$ can be expressed in a unique fashion as a linear combination of Hermite polynomials. By 2.2 and the orthogonality of the Hermite polynomials we see that the coefficient of $\alpha(K_{n-2k})$ in such an expression is just $p(G, k)$. ∎

From either of Corollaries 2.2 or 2.3 we see that the matchings polynomial of $G$ determines that of $\bar{G}$. Further, taking $x=0$ in 2.3 gives an explicit formula for $\alpha(\bar{G}, 0)$, and thus also for the number of perfect matchings in $\bar{G}$, in terms of the numbers $p(G, k)$. This formula is also derived, using inclusion-exclusion, in [8]. In this reference the multigraphs with the property that the matchings polynomial of each subgraph determines that of its complementary subgraph are determined.

**2.4. Theorem.** *Let G be a graph with n vertices. Then*

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty}\alpha(K_m, x)\alpha(G, x+y)\exp(-x^2/2)\,dx$$

$$=\begin{cases} m!\displaystyle\sum_{k=0}^{\left[\frac{n-m}{2}\right]}\binom{n-2k}{m}p(\bar{G}, k)y^{n-2k-m} & m\leqq n,\\ 0 & m>n.\end{cases}$$

**Proof.** It is straightforward to verify that $(n-2k)p(K_n, k)=np(K_{n-1}, k)$, whence it follows from the definition of $\alpha(G, x)$ that $\frac{d}{dx}\alpha(K_n)=n\alpha(K_{n-1})$. Hence the Taylor series expansion of $\alpha(K_n, x+y)$ in powers of $y$ can be written as

$$\alpha(K_n, x+y)=\sum_{j=0}^{n}\binom{n}{j}\alpha(K_{n-j}, x)y^j.$$

Using this we see that the theorem holds when $G=K_n$, since in this case the integral in the statement of the theorem equals $m!\binom{n}{m}y^{n-m}$. Given this, the general result follows at once using 2.3. ∎

We note one corollary of the preceding result.

**2.5. Corollary.** *Let G be a graph with n vertices. Then the number of matchings in* $\bar{G}$ *is*

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty}\alpha(G, x+1)\exp(-x^2/2)\,dx.$$

**Proof.** Take $y=1$, $m=0$ in 2.4. ∎

## 3. Rook polynomials

**3.1.** A result analogous to 1.2 can be established for rook polynomials. We comment briefly on this.

A subset of the squares of an $m\times n$ chessboard ($m\leqq n$) will be referred to as a board. If $B$ is a board then $r(B, k)$ is the number of ways of placing $k$ non-attacking rooks on the squares of $B$, i.e. the number of choosing $k$ squares of $B$ with no two in the same row or column.

We identify a board $B$ with a graph $G=G(B)$ as follows. The vertices of $G(B)$ are the rows and columns of the full $m\times n$ chessboard, a row and column are adjacent if the square in which they intersect lies in $B$. Given this identification it follows that $r(B, k)=p(G(B), k)$. Note that if $B$ is the full $m\times n$ chessboard then $G(B)$ is the complete bipartite graph $K_{m,n}$.

If $G$ is a subgraph of $K_{m,n}$ we define the *rook polynomial* $r(G, x)$ to be

$$\sum_{k=0}^{m} (-1)^k p(G, k) x^{m-k}.$$

Many properties of rook polynomials are established in the last two chapters of Riordan's book [7]. In particular we find that $(-1)^m r(K_m, x)$ is the *Laguerre polynomial* $L_m(x)$ and that, if $t \geqq 0$, $(-1)^m r(K_{m,m+t}, x)$ is the *generalized Laguerre polynomial* $L_m^t(x)$. For information on the Laguerre polynomials see [2]. We also warn the reader that our notation for rook polynomials differs slightly from that used by Riordan.

If $G$ is a subgraph of $K_{m,n}$ we use $G^*$ to denote the subgraph of $K_{m,n}$ formed by those edges not in $G$. Taking the proof of 1.2 as a guide, we obtain:

**3.2. Theorem.** *Let $G$ be a subgraph of $K_{m,n}$. Then the number of perfect matchings in $G^*$ is equal to*

$$\int_0^\infty r(G, x) e^{-x}\, dx. \quad \blacksquare$$

This result also occurs as Corollary 2.1 in [5]. We note that if $G$ and $H$ are subgraphs of $K_{m,m+s}$ and $K_{n,n+s}$ respectively, then $G \cup H$ is a subgraph of $K_{l,l}$ where $l = m+n+s$. It is routine to verify that the rook polynomial of $G \cup H$ is $r(G, x) r(H, x) x^s$. If $H = K_{n,n+s}$ then the number of perfect matchings in $(G \cup H)^*$ is just $n!(n+s)!\, p(G^*, m-n)$ when $n \leqq m$, and is zero otherwise. Hence:

**3.3. Corollary.** *Let $G$ be a subgraph of $K_{m,m+s}$. Then*

$$\int_0^\infty r(G, x) r(K_{n,n+s}, x) x^s e^{-x}\, dx = \begin{cases} n!(n+s)!\, p(G^*, m-n) & n \leqq m, \\ 0 & n > 0. \end{cases} \quad \blacksquare$$

It follows from 3.3 that $r(G, x)$ determines $r(G^*, x)$, as was first proved by Riordan (see equation (22) on page 179 of [7]). Taking $G = K_{m,m+s}$ in 3.3 also yields the well-known fact that the generalized Laguerre polynomials $L_m^s(x)$ $(m = 0, 1, \ldots)$ are orthogonal on the non-negative reals with respect to the weight function $x^s e^{-x}$.

Using arguments analogous to those employed in the proof of 2.3, we also obtain:

**3.4. Corollary.** *Let $G$ be a subgraph of $K_{m,m+s}$. Then*

$$r(G^*, x) = \sum_{i=0}^{m} p(G, m-l) r(K_{l,l+s}, x). \quad \blacksquare$$

In conclusion we remark that, if $G$ is a subgraph of $K_{m,n}$ $(m \leqq n)$, then $\alpha(G, x) = x^{n-m} r(G, x^2)$. As a special case of this we have

$$\alpha(K_{m,n}, x) = x^{n-m} r(K_{m,n}, x^2) = (-1)^m x^{n-m} L_m^{n-m}(x^2).$$

Furthermore $K_{m,n}$ is the complement of $K_m \cup K_n$. Consequently we may use the results of Section 2 to derive relations between the Hermite and generalized Laguerre polynomials. For example taking $G = K_m \cup K_n$ in 2.3 yields the expansion of $L_m^{n-m}(x^2)$ in terms of Hermite polynomicals.

**Note in proof.** I have recently seen an apparently unpublished manuscript [1] by Ruth Azor, J. Gillis and J. D. Victor. This contains a proof our Theorem 1.2 under the additional assumption that $G$ is a complete multipartite graph.

Also the observation that $\alpha(G, x)$ determines $\alpha(\bar{G}, x)$ is an immediate consequence of Exercise 5.18 (a) in L. Lovász [6].

## References


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